

ON THE LOCATION OF EIGENVALUES OF MATRIX POLYNOMIALS

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ABSTRACT. A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of the matrix polynomial $P(z)$ if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $P(\lambda)x = 0$. Note that each finite eigenvalue of $P(z)$ is a zero of the *characteristic polynomial* $\det(P(z))$. In this paper we establish some (upper and lower) bounds for eigenvalues of matrix polynomials based on the norm of their coefficient matrices and compare these bounds to those given by N.J. Higham and F. Tisseur [8].

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1. INTRODUCTION

Let $\mathbb{C}^{n \times n}$ denote the set of all $n \times n$ matrices whose entries in \mathbb{C} . For a *matrix polynomial* we mean the matrix-valued function of a complex variable of the form

$$P(z) = A_m z^m + \cdots + A_1 z + A_0, \quad (1.1)$$

where $A_i \in \mathbb{C}^{n \times n}$ for all $i = 0, \dots, m$. If $A_m \neq 0$, $P(z)$ is called a matrix polynomial of *degree* m . When $A_m = I$, the identity matrix, the matrix polynomial $P(z)$ is called *monic*.

A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of the matrix polynomial $P(z)$, if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $P(\lambda)x = 0$. Then the vector x is called, as usual, an *eigenvector* associated to the eigenvalue λ . Note that each finite eigenvalue of $P(z)$ is a root of the *characteristic polynomial* $\det(P(z))$.

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The *polynomial eigenvalue problem (PEP)* is to find an eigenvalue λ and a non-zero vector $x \in \mathbb{C}^n$ such that $P(\lambda)x = 0$. For $m = 1$, (PEP) is actually the *generalized eigenvalue problem (GEP)*

$$Ax = \lambda Bx,$$

and, in addition, if $A_1 = I$, we have the standard eigenvalue problem

$$Ax = \lambda x.$$

For $m = 2$ we have the *quadratic eigenvalue problem (QEP)*.

The theory of matrix polynomials was primarily devoted by two works, both of which are strongly motivated by the theory of *vibrating systems*: one by Frazer, Duncan, and Collar in 1938 [FDC], and the other by P. Lancaster in 1966 [10].

(QEPs), and more generally (PEPs), play an important role in applications to science and engineering. We refer to [17] for a survey on applications of (QEP). Moreover, we refer to the book of I. Gohberg, P. Lancaster and L. Rodman [6] for a theory of matrix polynomials and their applications.

There are algorithms to solve (QEPs), see the works of Hamarling, Munro and Tisseur [7, 2013] and Zeng and Su [18, 2014]. For (PEPs), there are some researchs on bounds for eigenvalues of matrix polynomials which were constructed in terms of norms of coefficients of the given matrix polynomials. See, for example, the work of Higham and Tisseur [8, 2003].

Computing eigenvalues of matrix polynomials (even computing eigenvalues of scalar matrices and finding roots of univariate polynomials) is a hard problem. There is an iterative method to compute these eigenvalues, see Simoncini and Perotti [15, 2006]. Moreover, when computing pseudospectra of matrix polynomials, which provide information about the global sensitivity of the eigenvalues, a particular region of the (possibly extended) complex plane must be identified that contains the eigenvalues of interest, and bounds clearly help to determine such region [16]. Therefore, it is useful to find the location of these eigenvalues. Note that, if A_m is singular, then $P(z)$ has an infinite eigenvalue, and if A_0 is singular then 0 is an eigenvalue of $P(z)$. Thus throughout this paper *we always assume A_m and A_0 to be non-singular* in order to find upper and lower bounds for the magnitude of eigenvalues.

N.J. Higham and F. Tisseur [8] have given some bounds for eigenvalues of matrix polynomials based on the norm of their coefficient matrices. Continuing the idea of N.J. Higham and F. Tisseur, in this paper we establish some other bounds for the magnitude of eigenvalues of the matrix polynomial $P(z)$, and compare these bounds to those given by them.

The paper is organized as follows. In Section 2 we give bounds for matrix polynomials whose coefficients satisfy some special properties, in particular, we give a matrix version of Eneström-Kakeya's theorem. In Section 3 we establish matrix versions of some Cauchy's type theorems. In particular, we establish a matrix version of the theorem of Joyal, Labelle and Rahman (cf. [9], [12, Theorem 2.14]) and some of its corollaries. Moreover, we give also a

matrix version of Datt and Govil's theorem [3, Theorem 1] and some other bounds. Finally, we give some numerical experiments in Section 4.

Notation. For a matrix $A \in \mathbb{C}^{n \times n}$, the notation $A \geq 0$ means " A is positive semidefinite", i.e. for every vector $x \in \mathbb{C}^n$ we have $x^*Ax \geq 0$; $A > 0$ means " A is positive definite", i.e. $x^*Ax > 0$ for every nonzero vector $x \in \mathbb{C}^n$. For two matrices $A, B \in \mathbb{C}^{n \times n}$, the notation $A \geq B$ means $A - B \geq 0$.

Throughout this paper $\|\cdot\|$ denotes for a subordinate matrix norm.

2. ENESTRÖM-KAKEYA'S THEOREM FOR MATRIX POLYNOMIALS

In this section we give upper and lower bounds for eigenvalues of some special matrix polynomials. First of all we consider matrix polynomials with dominant property.

Theorem 2.1. *Let $P(z) = A_0 + A_1z + \cdots + A_mz^m$ be a matrix polynomial whose coefficients $A_i \in \mathbb{C}^{n \times n}$ satisfying the following dominant property:*

$$\|A_m\| > \|A_i\|, \forall i = 0, \dots, m-1.$$

Then all eigenvalues λ of $P(z)$ locate in the open disk

$$|\lambda| < 1 + \|A_m\| \|A_m^{-1}\|.$$

In particular, for $n = 1$, we obtain the following corollary of Cauchy's theorem ([11, Theorem (27,2)]; see also [2, Theorem 2.2]): *Let $p(z) = a_0 + a_1z + \cdots + a_mz^m \in \mathbb{C}[z]$ such that $|a_m| > |a_i|$ for all $i = 0, \dots, m-1$. Then all the roots of $p(z)$ locate in the open disk $|z| < 2$. The proof of this corollary uses the fact that when $n = 1$ we have $\|A_m\| \|A_m^{-1}\| = 1$.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector associated to λ .

We have nothing to prove if $|\lambda| \leq 1$. Hence we may assume that $|\lambda| > 1$.

Then we have

$$\begin{aligned}
\|P(\lambda)x\| &\geq |\lambda|^m \left[\|A_m x\| - \left\| \sum_{i=0}^{m-1} \frac{A_i x}{\lambda^{m-i}} \right\| \right] \\
&\geq |\lambda|^m \left[\|A_m^{-1}\|^{-1} - \sum_{i=0}^{m-1} \frac{\|A_i\|}{|\lambda|^{m-i}} \right] \\
&\geq |\lambda|^m \left[\|A_m^{-1}\|^{-1} - \sum_{i=0}^{m-1} \frac{\|A_m\|}{|\lambda|^{m-i}} \right] \\
&= |\lambda|^m \|A_m^{-1}\|^{-1} \left[1 - \|A_m\| \|A_m^{-1}\| \sum_{i=1}^m \frac{1}{|\lambda|^i} \right] \\
&> |\lambda|^m \|A_m^{-1}\|^{-1} \left[1 - \|A_m\| \|A_m^{-1}\| \sum_{i=1}^{\infty} \frac{1}{|\lambda|^i} \right] \\
&= |\lambda|^m \|A_m^{-1}\|^{-1} \left[1 - \frac{\|A_m\| \|A_m^{-1}\|}{|\lambda| - 1} \right] \\
&= \frac{|\lambda|^m \|A_m^{-1}\|^{-1}}{|\lambda| - 1} (|\lambda| - 1 - \|A_m\| \|A_m^{-1}\|).
\end{aligned}$$

Hence, if $|\lambda| \geq 1 + \|A_m\| \|A_m^{-1}\|$ we have $\|P(\lambda)x\| > 0$, a contradiction. It follows that $|\lambda| < 1 + \|A_m\| \|A_m^{-1}\|$, which completes the proof. \square

We know the following well-known Eneström-Kakeya's theorem for polynomials.

Theorem 2.2 ([14, Corollary 3]). *Let $p(z)$ be a polynomial in one variable given by*

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m, \quad a_i \in \mathbb{R}, \forall i = 1, \dots, m.$$

Suppose that

$$a_m \geq a_{m-1} \geq \cdots \geq a_0 \geq 0; \quad a_m > 0.$$

If $z \in \mathbb{C}$ is a root of $p(z)$ then $\frac{a_0}{2a_m} \leq |z| \leq 1$.

A matrix version of Theorem 2.2 can be given as follows.

Theorem 2.3. *Let $P(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a matrix polynomial whose coefficients $A_i \in \mathbb{C}^{n \times n}$ satisfying*

$$A_m \geq A_{m-1} \geq \cdots \geq A_0 \geq 0; \quad A_m > 0.$$

Then each eigenvalue λ of $P(z)$ satisfies

$$\frac{\lambda_{\min}(A_0)}{2\lambda_{\max}(A_m)} \leq |\lambda| \leq 1,$$

where $\lambda_{\min}(A_0)$ denotes for the smallest eigenvalue of A_0 and $\lambda_{\max}(A_m)$ denotes for the largest eigenvalue of A_m .

Proof. A proof for the upper bound of $|\lambda|$ in this theorem was given by G. Dirr and H. K. Wimmer [4, Theorem 2.1]. Now we give a proof for the lower bound.

Firstly we observe that for a matrix $A \in \mathbb{C}^{n \times n}$, its smallest eigenvalue $\lambda_{\min}(A)$ and its largest eigenvalue $\lambda_{\max}(A)$ belong to the set

$$\{x^*Ax | x \in \mathbb{C}^n, \|x\| = 1\}.$$

Hence for a unit vector $x \in \mathbb{C}^n$, we always have

$$\lambda_{\min}(A) \leq x^*Ax \leq \lambda_{\max}(A). \quad (2.1)$$

Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$, and $u \in \mathbb{C}^n, \|u\| = 1$ an eigenvector associated to λ . Consider the polynomial

$$P_u(z) := u^*P(z)u = \sum_{i=0}^m (u^*A_i u) z^i.$$

Note that λ is a root of $P_u(z)$. Moreover, the hypothesis on the relation of A_i 's implies that

$$u^*A_m u \geq u^*A_{m-1}u \geq \cdots \geq u^*A_0 u \geq 0, u^*A_m u > 0,$$

that is, the polynomial $P_u(z)$ satisfies the conditions given in Theorem 2.2. Applying this theorem for $P_u(z)$ we obtain

$$\frac{u^*A_0 u}{2u^*A_m u} \leq |\lambda|.$$

Then the required lower bound for $|\lambda|$ follows from (2.1). \square

By applying Theorem 2.3 for the matrix polynomial $z^n P(\frac{1}{z})$, $z \neq 0$, we obtain the following dual version of Theorem 2.3.

Theorem 2.4. *Let $P(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a matrix polynomial whose coefficients $A_i \in \mathbb{C}^{n \times n}$ satisfying*

$$A_0 \geq A_1 \geq \cdots \geq A_m > 0.$$

Then for each eigenvalue λ of $P(z)$ we have $|\lambda| \geq 1$.

We have also the following version of Eneström-Kakeya's theorem for polynomials.

Theorem 2.5 (Eneström-Kakeya's theorem, Version 2, [1]). *Let $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ be a polynomial whose coefficients $a_i, i = 0, \dots, m$ are positive real numbers. Denote*

$$\alpha := \min_{0 \leq i \leq m-1} \left\{ \frac{a_i}{a_{i+1}} \right\}, \beta := \max_{0 \leq i \leq m-1} \left\{ \frac{a_i}{a_{i+1}} \right\}.$$

Then for a root $z \in \mathbb{C}$ of $p(z)$, we have

$$\alpha \leq |z| \leq \beta.$$

Using the same method as given in the proof of Theorem 2.3, applying Theorem 2.5, we obtain the following bounds for eigenvalues of matrix polynomials whose coefficients are positive definite.

Theorem 2.6. *Let $P(z) = A_0 + A_1z + \cdots + A_mz^m$ be a matrix polynomial whose coefficients $A_i \in \mathbb{C}^{n \times n}$ are positive definite. If $\lambda \in \mathbb{C}$ is an eigenvalue of $P(z)$, then*

$$\min_{i=0, \dots, m-1} \left\{ \frac{\lambda_{\min}(A_i)}{\lambda_{\max}(A_{i+1})} \right\} \leq |\lambda| \leq \max_{i=0, \dots, m-1} \left\{ \frac{\lambda_{\max}(A_i)}{\lambda_{\min}(A_{i+1})} \right\}.$$

3. CAUCHY TYPE THEOREMS FOR MATRIX POLYNOMIALS

Theorem 3.1 (Cauchy's theorem for matrix polynomials, [8, Lemma 3.1]). *Let $P(z) = A_0 + A_1z + \cdots + A_mz^m$ be a matrix polynomial. Let r resp. R be the positive root of the polynomial*

$$h(z) = z^m \|A_m\| + z^{m-1} \|A_{m-1}\| + \cdots + z \|A_1\| - \|A_0^{-1}\|^{-1}$$

resp.

$$g(z) = z^m \|A_m^{-1}\|^{-1} - z^{m-1} \|A_{m-1}\| - \cdots - \|A_0\|.$$

Then all eigenvalues λ of $P(z)$ satisfy

$$r \leq |\lambda| \leq R.$$

Theorem 3.2. *Let $P(z) = A_0 + A_1z + \cdots + A_mz^m$ be a matrix polynomial. Denote*

$$M := \|A_m^{-1}\| \max_{i=0, \dots, m-1} \|A_i\|.$$

Then all eigenvalues of $P(z)$ are contained in the closed disk

$$K(0, r_1) := \{z \in \mathbb{C} \mid |z| \leq r_1\},$$

where $r_1 := \max\{1, \delta\}$ and $\delta \neq 1$ is the positive root of the equation

$$z^{m+1} - (1 + M)z^m + M = 0.$$

In particular, for $n = 1$ we obtain a Cauchy type theorem for polynomials [2, Theorem 3.2].

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector associated to λ .

The conclusion is clear if $|\lambda| \leq 1$. Therefore we may assume that $|\lambda| > 1$.

Then we have

$$\|P(\lambda)x\| \geq \left[\|A_mx\| |\lambda|^m - \left\| \sum_{i=0}^{m-1} A_i x \lambda^i \right\| \right] \quad (3.1)$$

$$\geq \|A_m^{-1}\|^{-1} \left[|\lambda|^m - \sum_{i=0}^{m-1} \|A_i\| \|A_m^{-1}\| |\lambda|^i \right] \quad (3.2)$$

$$\begin{aligned} &\geq \|A_m^{-1}\|^{-1} \left[|\lambda|^m - M \sum_{i=0}^{m-1} |\lambda|^i \right] \quad (3.3) \\ &= \|A_m^{-1}\|^{-1} \left[|\lambda|^m - M \frac{|\lambda|^m - 1}{|\lambda| - 1} \right] \\ &= \frac{\|A_m^{-1}\|^{-1}}{|\lambda| - 1} (|\lambda|^{m+1} - (1+M)|\lambda|^m + M). \end{aligned}$$

In the lines above, from (3.1) to (3.2) we use the fact that $\|A_m\| \geq \|A_m^{-1}\|^{-1}$; from (3.2) to (3.3) we use the definition of M .

Note that the polynomial $f(z) := z^{m+1} - (1+M)z^m + M$ has exactly two positive real roots 1 and $\delta \neq 1$ by the Descartes's rule of signs, and $f(0) > 0$. It follows that

$$|f(z)| > 0 \text{ for all } z > \max\{\delta, 1\}.$$

Hence for $|\lambda| > r_1$ we have $\|P(\lambda)x\| > 0$, a contradiction. This completes the proof. \square

Corollary 3.3. *Let $P(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a matrix polynomial. Denote*

$$\widetilde{M} := \|A_m^{-1}\| \max_{i=0, \dots, m} \|A_{m-i} - A_{m-i-1}\|.$$

Then all eigenvalues of $P(z)$ are contained in the closed disk $K(0, r_2)$, where $r_2 := \max\{1, \delta\}$ and $\delta \neq 1$ is the positive root of the equation

$$z^{m+2} - (1 + \widetilde{M})z^{m+1} + \widetilde{M} = 0.$$

In particular, for $n = 1$ we obtain [2, Theorem 3.3].

Proof. Consider the matrix polynomial

$$Q(z) := (1 - z)P(z) = -A_m z^{m+1} + \sum_{i=0}^m (A_{m-i} - A_{m-i-1}) z^{m-i}.$$

Applying Theorem 3.2 for the polynomial $Q(z)$, observing that each eigenvalue of $P(z)$ is also an eigenvalue of $Q(z)$, we obtain the required result. \square

Theorem 3.4. *Let $P(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a matrix polynomial. Then all eigenvalues of $P(z)$ are contained in the open disk*

$$K^o(0, r_3) := \{z \in \mathbb{C} \mid |z| < r_3\},$$

where $r_3 := 1 + M$ and M is defined as in Theorem 3.2.

In particular, for $n = 1$ we obtain another Cauchy's theorem for polynomials [11, Theorem (27,2)].

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector associated to λ .

As above, we may assume that $|\lambda| > 1$. Then we have

$$\begin{aligned}
\|P(\lambda)x\| &\geq |\lambda|^m \left[\|A_m x\| - \left\| \sum_{i=0}^{m-1} \frac{A_i x}{\lambda^{m-i}} \right\| \right] \\
&\geq |\lambda|^m \|A_m^{-1}\|^{-1} \left[1 - \|A_m^{-1}\| \sum_{i=0}^{m-1} \frac{\|A_i\|}{|\lambda|^{m-i}} \right] \\
&\geq |\lambda|^m \|A_m^{-1}\|^{-1} \left[1 - M \sum_{i=1}^m \frac{1}{|\lambda|^i} \right] \\
&> |\lambda|^m \|A_m^{-1}\|^{-1} \left[1 - M \sum_{i=1}^{\infty} \frac{1}{|\lambda|^i} \right] \\
&= |\lambda|^m \|A_m^{-1}\|^{-1} \left[1 - \frac{M}{|\lambda| - 1} \right] \\
&= \frac{|\lambda|^m \|A_m^{-1}\|^{-1}}{|\lambda| - 1} (|\lambda| - 1 - M).
\end{aligned}$$

Then, for $|\lambda| \geq 1 + M$ we have $\|P(\lambda)x\| > 0$, a contradiction. Thus $|\lambda| < 1 + M$. \square

Corollary 3.5. *Let $P(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a matrix polynomial. Then all eigenvalues of $P(z)$ are contained in the open disk $K^o(0, r_4)$, where $r_4 := 1 + \widetilde{M}$ and \widetilde{M} is defined as in Corollary 3.3.*

In particular, for $n = 1$ we obtain [2, Theorem 3.4].

Proof. Consider the matrix polynomial

$$Q(z) := (1 - z)P(z) = -A_m z^{m+1} + \sum_{i=0}^m (A_{m-i} - A_{m-i-1}) z^{m-i}.$$

Since each eigenvalue of $P(z)$ is one of that of $Q(z)$, applying Theorem 3.4 for $Q(z)$ we have the conclusion. \square

Next we give a matrix version of the theorem of Joyal, Labelle and Rahman, cf. [9], [12, Theorem 2.14].

Theorem 3.6. *Let $P(z) = A_0 + A_1 z + \cdots + I \cdot z^m$ be a matrix polynomial. Denote*

$$\alpha := \max_{i=0, \dots, m-2} \|A_i\|.$$

Then for each eigenvalue λ of $P(z)$ we have

$$|\lambda| \leq \frac{1}{2} \left\{ 1 + \|A_{m-1}\| + \left[(1 - \|A_{m-1}\|)^2 + 4\alpha \right]^{\frac{1}{2}} \right\}.$$

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector associated to λ .

By contradiction, assume

$$|\lambda| > \frac{1}{2} \left\{ 1 + \|A_{m-1}\| + \left[(1 - \|A_{m-1}\|)^2 + 4\alpha \right]^{\frac{1}{2}} \right\}.$$

It follows that

$$(|\lambda| - 1)(|\lambda| - \|A_{m-1}\|) - \alpha > 0. \quad (3.4)$$

Multiplying (3.4) by $|\lambda|^{m-1}$ and then dividing by $|\lambda| - 1$, we obtain

$$|\lambda|^m - \|A_{m-1}\| \lambda^{m-1} - \alpha \frac{|\lambda|^{m-1}}{|\lambda| - 1} > 0.$$

However,

$$\begin{aligned} \alpha \frac{|\lambda|^{m-1}}{|\lambda| - 1} &> \alpha \frac{|\lambda|^{m-1} - 1}{|\lambda| - 1} = \alpha(1 + |\lambda| + \cdots + |\lambda|^{m-2}) \\ &\geq \|(A_0 + A_1\lambda + \cdots + A_{m-2}\lambda^{m-2})x\|. \end{aligned}$$

On the other hand,

$$|\lambda|^m - \|A_{m-1}\| \lambda^{m-1} \leq \|(I \cdot \lambda^m + A_{m-1}\lambda^{m-1})x\|.$$

It follows that

$$\begin{aligned} 0 &< |\lambda|^m - \|A_{m-1}\| \lambda^{m-1} - \alpha \frac{|\lambda|^{m-1}}{|\lambda| - 1} \\ &< \|(I \cdot \lambda^m + A_{m-1}\lambda^{m-1})x\| - \|(A_0 + A_1\lambda + \cdots + A_{m-2}\lambda^{m-2})x\| \\ &\leq \|(A_0 + A_1\lambda + \cdots + A_{m-2}\lambda^{m-2})x + (A_{m-1}\lambda^{m-1} + I \cdot \lambda^m)x\| = \|P(\lambda)x\|, \end{aligned}$$

a contradiction. Thus

$$\lambda \leq \frac{1}{2} \left\{ 1 + \|A_{m-1}\| + \left[(1 - \|A_{m-1}\|)^2 + 4\alpha \right]^{\frac{1}{2}} \right\}.$$

□

This theorem gives the following upper bound for a general matrix polynomial.

Corollary 3.7. *Let $P(z) = A_0 + A_1z + \cdots + A_mz^m$ be a matrix polynomial.*

Denote

$$\alpha' := \max_{i=0, \dots, m-2} \|A_i A_m^{-1}\|.$$

Then for each eigenvalue λ of $P(z)$ we have

$$|\lambda| \leq \frac{1}{2} \left\{ 1 + \|A_{m-1} A_m^{-1}\| + \left[(1 - \|A_{m-1} A_m^{-1}\|)^2 + 4\alpha' \right]^{\frac{1}{2}} \right\}.$$

By applying Corollary 3.7 for the matrix polynomial $z^m P(\frac{1}{z})$, $z \neq 0$, we get the following lower bound for eigenvalues of $P(z)$.

Corollary 3.8. *Let $P(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a matrix polynomial. Denote*

$$\beta := \max_{i=2, \dots, m} \|A_i A_0^{-1}\|.$$

Then for each eigenvalue λ of $P(z)$ we have

$$|\lambda| \geq \frac{2}{1 + \|A_1 A_0^{-1}\| + \left[(1 - \|A_1 A_0^{-1}\|)^2 + 4\beta \right]^{\frac{1}{2}}}.$$

By applying Corollary 3.7 for the matrix polynomial $(1-z)P(z)$ we obtain

Corollary 3.9. *Let $P(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a matrix polynomial. Denote*

$$\gamma := \max_{i=1, \dots, m} \|(A_{m-i} - A_{m-i-1})A_m^{-1}\| \quad (A_{-1} = 0).$$

Then for each eigenvalue λ of $P(z)$ we have

$$|\lambda| \leq \frac{1}{2} \left\{ 1 + \|(A_m - A_{m-1})A_m^{-1}\| + \left[(1 - \|(A_m - A_{m-1})A_m^{-1}\|)^2 + 4\gamma \right]^{\frac{1}{2}} \right\}.$$

Similarly, Corollary 3.9 yields the following lower bound.

Corollary 3.10. *Let $P(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a matrix polynomial. Denote*

$$\gamma' := \max_{i=1, \dots, m} \|(A_i - A_{i+1})A_0^{-1}\| \quad (A_{m+1} = 0).$$

Then each eigenvalue λ of $P(z)$ satisfies

$$|\lambda| \geq \frac{2}{1 + \|(A_0 - A_1)A_0^{-1}\| + \left[(1 - \|(A_0 - A_1)A_0^{-1}\|)^2 + 4\gamma' \right]^{\frac{1}{2}}}.$$

By applying Theorem 3.6 for the matrix polynomial $(I \cdot z - A_{m-1})P(z)$ we obtain

Corollary 3.11. *Let $P(z) = A_0 + A_1 z + \cdots + I \cdot z^m$ be a matrix polynomial. Denote*

$$\delta := \max_{i=0, \dots, m-1} \|A_{m-1} A_i - A_{i-1}\| \quad (A_{-1} = 0).$$

Then for each eigenvalue λ of $P(z)$ we have

$$|\lambda| \leq \frac{1}{2}(1 + \sqrt{1 + 4\delta}).$$

This corollary gives the following upper bound for a general matrix polynomial.

Corollary 3.12. *Let $P(z) = A_0 + A_1z + \cdots + A_mz^m$ be a matrix polynomial. Denote*

$$\delta' := \max_{i=0, \dots, m-1} \left\| ((A_{m-1}A_m^{-1})A_i - A_{i-1})A_m^{-1} \right\| \quad (A_{-1} = 0).$$

Then for each eigenvalue λ of $P(z)$ we have

$$|\lambda| \leq \frac{1}{2}(1 + \sqrt{1 + 4\delta'}).$$

Corollary 3.12 yields the following lower bound.

Corollary 3.13. *Let $P(z) = A_0 + A_1z + \cdots + A_mz^m$ be a matrix polynomial. Denote*

$$\delta'' := \max_{i=1, \dots, m} \left\| ((A_1A_0^{-1})A_i - A_{i+1})A_0^{-1} \right\| \quad (A_{m+1} = 0).$$

Then for each eigenvalue λ of $P(z)$ we have

$$|\lambda| \geq \frac{2}{1 + \sqrt{1 + 4\delta''}}.$$

By applying Theorem 3.6 for the matrix polynomial $(I \cdot z + I - A_{m-1})P(z)$ we obtain

Corollary 3.14. *Let $P(z) = A_0 + A_1z + \cdots + I \cdot z^m$ be a matrix polynomial. Denote*

$$\epsilon := \max_{i=0, \dots, m-1} \left\| (I - A_{m-1})A_i + A_{i-1} \right\| \quad (A_{-1} = 0).$$

Then for each eigenvalue λ of $P(z)$ we have

$$|\lambda| \leq 1 + \sqrt{\epsilon}.$$

This yields the following upper bound for a general matrix polynomial.

Corollary 3.15. *Let $P(z) = A_0 + A_1z + \cdots + A_mz^m$ be a matrix polynomial. Denote*

$$\epsilon' := \max_{i=0, \dots, m-1} \left\| \left((I - A_{m-1}A_m^{-1})A_i + A_{i-1} \right) A_m^{-1} \right\| \quad (A_{-1} = 0).$$

Then for each eigenvalue λ of $P(z)$ we have

$$|\lambda| \leq 1 + \sqrt{\epsilon'}.$$

The following lower bound is obtained by applying Corollary 3.15 for the matrix polynomial $z^m P(\frac{1}{z}), z \neq 0$.

Corollary 3.16. *Let $P(z) = A_0 + A_1z + \cdots + A_mz^m$ be a matrix polynomial. Denote*

$$\epsilon'' := \max_{i=1, \dots, m} \left\| \left((I - A_1A_0^{-1})A_i + A_{i+1} \right) A_0^{-1} \right\| \quad (A_{m+1} = 0).$$

Then for each eigenvalue λ of $P(z)$ we have

$$|\lambda| \geq \frac{1}{1 + \sqrt{\epsilon''}}.$$

Next we give the matrix version of the theorem of Datt and Govil [3, Theorem 1].

Theorem 3.17. *Let $P(z) = A_0 + A_1z + \cdots + I \cdot z^m$ be a matrix polynomial. Denote*

$$A = \max_{i=0, \dots, m-1} \|A_i\|.$$

Then for each eigenvalue λ of $P(z)$ we have

$$\frac{\|A_0^{-1}\|^{-1}}{2(1+A)^{m-1}(Am+1)} \leq |\lambda| \leq 1 + \lambda_0 A,$$

where λ_0 is a root of the equation $x = 1 - \frac{1}{(Ax+1)^m}$ in the interval $(0, 1)$.

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector associated to λ .

First we prove the upper bound for $|\lambda|$. We consider two cases:

The first case: $mA \leq 1$. In this case, if $|\lambda| > 1$, we have

$$\|P(\lambda)x\| \geq |\lambda|^m - mA|\lambda|^{m-1} \geq |\lambda|^m - |\lambda|^{m-1} > 0, \text{ a contradiction.}$$

It follows that $|\lambda| \leq 1 \leq 1 + \lambda_0 A$ for all $\lambda_0 \in (0, 1)$.

The second case: $mA > 1$. In this case the equation $x = 1 - \frac{1}{(Ax+1)^m}$ has a unique root $\lambda_0 \in (0, 1)$ [3, Lemma 2]. Moreover, we have

$$\|P(\lambda)x\| \geq |\lambda|^m - A \sum_{j=0}^{m-1} |\lambda|^j = |\lambda|^m - A \frac{|\lambda|^m - 1}{|\lambda| - 1}.$$

If $|\lambda| > 1 + A\lambda_0$, we can write $|\lambda| = 1 + A\alpha$ with $\alpha > \lambda_0$. Then $\alpha > 1 - \frac{1}{(A\alpha+1)^m}$. It follows that

$$\|P(\lambda)x\| \geq (1 + A\alpha)^m - \frac{(1 + A\alpha)^m - 1}{\alpha} > 0,$$

a contradiction. Thus $|\lambda| \leq 1 + A\lambda_0$.

Now we prove the lower bound for $|\lambda|$. By contradiction, assume $|\lambda| < \frac{\|A_0^{-1}\|^{-1}}{2(1+A)^{m-1}(Am+1)}$. Let us consider the matrix polynomial $G(z) := (1-z)P(z)$. We have

$$G(z) = A_0 + \sum_{i=1}^m (A_i - A_{i-1})z^i + I \cdot z^m - A_{m-1}z^m - I \cdot z^{m+1} =: A_0 + H(z).$$

Denote $R := 1 + A$. Then for $|z| = R$, we have

$$\begin{aligned} \max_{|z|=R} \|H(z)x\| &\leq R^{m+1} + R^m + \|A_{m-1}\| R^m + \sum_{i=1}^{m-1} \|A_i - A_{i-1}\| R^i \\ &\leq R^m [R + 1 + A + 2(m-1)A] \\ &= 2(1+A)^m (mA + 1). \end{aligned}$$

It follows from the *maximal module principle* that for $|z| \leq R$ we have

$$\|H(z)x\| \leq 2(1+A)^m(mA+1).$$

Then for $|\lambda| < \frac{\|A_0^{-1}\|^{-1}}{2(1+A)^{m-1}(Am+1)} < R$ we have

$$\begin{aligned} \|G(\lambda)x\| &= \|A_0x + H(\lambda)x\| \geq \|A_0^{-1}\|^{-1} - \|H(\lambda)x\| \\ &\geq \|A_0^{-1}\|^{-1} - \frac{|\lambda|}{1+A} \max_{|\lambda| \leq 1+A} \|H(\lambda)x\| \\ &\geq \|A_0^{-1}\|^{-1} - 2(1+A)^{m-1}(mA+1)|\lambda| > 0, \end{aligned}$$

a contradiction. Therefore

$$\frac{\|A_0^{-1}\|^{-1}}{2(1+A)^{m-1}(Am+1)} \leq |\lambda|.$$

□

This result gives the following bounds for general matrix polynomials.

Corollary 3.18. *Let $P(z) = A_0 + A_1z + \dots + A_mz^m$ be a matrix polynomial. Denote*

$$A' = \max_{i=0, \dots, m-1} \|A_i A_m^{-1}\|.$$

Then for each eigenvalue λ of $P(z)$ we have

$$\frac{\|A_m A_0^{-1}\|^{-1}}{2(1+A')^{m-1}(A'm+1)} \leq |\lambda| \leq 1 + \lambda_0 A',$$

where λ_0 is a root of the equation $x = 1 - \frac{1}{(A'x+1)^m}$ in the interval $(0, 1)$.

If we do not wish to look for a root in the interval $(0, 1)$ of the equation $x = 1 - \frac{1}{(A'x+1)^m}$, we use the following upper bound.

Corollary 3.19. *Let $P(z) = A_0 + A_1z + \dots + A_mz^m$ be a matrix polynomial. Denote*

$$A' = \max_{i=0, \dots, m-1} \|A_i A_m^{-1}\|.$$

Then for each eigenvalue λ of $P(z)$ we have

$$\frac{\|A_m A_0^{-1}\|^{-1}}{2(1+A')^{m-1}(A'm+1)} \leq |\lambda| < 1 + \left(1 - \frac{1}{(1+A')^m}\right) A'.$$

Proof. The proof follows from Corollary 3.18 and the fact that for a root λ_0 of the equation $x = 1 - \frac{1}{(A'x+1)^m}$ in the interval $(0, 1)$, we have always

$$\lambda_0 < 1 - \frac{1}{(1+A')^m}.$$

□

Next we give some other bounds for the magnitude of eigenvalues of matrix polynomials.

Theorem 3.20. *Let $P(z) = A_0 + A_1z + \cdots + A_mz^m$ be a matrix polynomial. Denote*

$$M := \max_{i=0, \dots, m-1} \|A_i\|, \quad M' := \max_{i=1, \dots, m} \|A_i\|.$$

Then all eigenvalues λ of $P(z)$ satisfy

$$\frac{\|A_0^{-1}\|^{-1}}{\|A_0^{-1}\|^{-1} + M'} < |\lambda| < 1 + \frac{M}{\|A_m^{-1}\|^{-1}}.$$

In particular, for $n = 1$ we obtain [12, Theorem 2.2].

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector associated to λ .

If $|\lambda| \geq 1 + \frac{M}{\|A_m^{-1}\|^{-1}}$, we have

$$\begin{aligned} \|P(\lambda)x\| &\geq |\lambda|^m \|A_mx\| - \|A_{m-1}\lambda^{m-1}x + \cdots + A_0x\| \\ &\geq \|A_m^{-1}\|^{-1} |\lambda|^m - \sum_{i=0}^{m-1} \|A_i\| |\lambda|^i \\ &\geq \|A_m^{-1}\|^{-1} |\lambda|^m - M \sum_{i=0}^{m-1} |\lambda|^i \\ &= \|A_m^{-1}\|^{-1} |\lambda|^m \left(1 - \frac{M}{\|A_m^{-1}\|^{-1} |\lambda|^m} \sum_{i=0}^{m-1} |\lambda|^i \right) \\ &= \|A_m^{-1}\|^{-1} |\lambda|^m \left(1 - \frac{M}{\|A_m^{-1}\|^{-1}} \sum_{i=1}^m \frac{1}{|\lambda|^i} \right) \\ &> \|A_m^{-1}\|^{-1} |\lambda|^m \left(1 - \frac{M}{\|A_m^{-1}\|^{-1}} \sum_{i=1}^{\infty} \frac{1}{|\lambda|^i} \right) \\ &= \|A_m^{-1}\|^{-1} |\lambda|^m \left(1 - \frac{M}{\|A_m^{-1}\|^{-1}} \frac{1}{|\lambda| - 1} \right) \geq 0, \text{ a contradiction.} \end{aligned}$$

Similarly, if $|\lambda| \leq \frac{\|A_0^{-1}\|^{-1}}{\|A_0^{-1}\|^{-1} + M}$, we have

$$\begin{aligned}
\|P(\lambda)x\| &\geq \|A_0^{-1}\|^{-1} - \sum_{i=1}^m |\lambda|^i \|A_i\| \\
&\geq \|A_0^{-1}\|^{-1} - M' \sum_{i=1}^m |\lambda|^i \\
&> \|A_0^{-1}\|^{-1} - M' \frac{|\lambda|}{1 - |\lambda|} \\
&= \frac{\|A_0^{-1}\|^{-1} (1 - |\lambda|) - M' |\lambda|}{1 - |\lambda|} \geq 0, \text{ a contradiction.}
\end{aligned}$$

It follows that $|\lambda| > \frac{\|A_0^{-1}\|^{-1}}{\|A_0^{-1}\|^{-1} + M}$. This completes the proof. \square

More generally, we have the following bounds.

Theorem 3.21. *Let $P(z) = A_0 + A_1 z + \dots + A_m z^m$ be a matrix polynomial. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Denote*

$$M_p := \left(\sum_{i=0}^{m-1} \|A_i\|^p \right)^{\frac{1}{p}}, \quad M'_p := \left(\sum_{i=1}^m \|A_i\|^p \right)^{\frac{1}{p}}.$$

Then for an eigenvalue λ of $P(z)$ we have

$$\left[\frac{\|A_0^{-1}\|^{-q}}{(M'_p)^q + \|A_0^{-1}\|^{-q}} \right]^{\frac{1}{q}} < |\lambda| < \left[1 + \left(\frac{M_p}{\|A_m^{-1}\|^{-1}} \right)^q \right]^{\frac{1}{q}}.$$

In particular, for $n = 1$ we obtain [12, Theorem 2.4]. Moreover, letting p tend to infinity (then q tends to 1), we obtain Theorem 3.20.

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector associated to λ .

If $|\lambda| \geq \left[1 + \left(\frac{M_p}{\|A_m^{-1}\|^{-1}}\right)^q\right]^{\frac{1}{q}}$, we have

$$\|P(\lambda)x\| \geq \|A_m^{-1}\|^{-1} |\lambda|^m - \sum_{i=0}^{m-1} \|A_i\| |\lambda|^i \quad (3.5)$$

$$\geq \|A_m^{-1}\|^{-1} |\lambda|^m - \left(\sum_{i=0}^{m-1} \|A_i\|^p\right)^{\frac{1}{p}} \left(\sum_{i=0}^{m-1} |\lambda|^{iq}\right)^{\frac{1}{q}} \quad (3.6)$$

$$\begin{aligned} &= \|A_m^{-1}\|^{-1} |\lambda|^m \left[1 - \frac{M_p}{\|A_m^{-1}\|^{-1} |\lambda|^m} \left(\sum_{i=0}^{m-1} |\lambda|^{iq}\right)^{\frac{1}{q}}\right] \\ &= \|A_m^{-1}\|^{-1} |\lambda|^m \left[1 - \frac{M_p}{\|A_m^{-1}\|^{-1}} \left(\sum_{i=0}^{m-1} |\lambda|^{(i-m)q}\right)^{\frac{1}{q}}\right] \\ &> \|A_m^{-1}\|^{-1} |\lambda|^m \left[1 - \frac{M_p}{\|A_m^{-1}\|^{-1}} \left(\sum_{i=1}^{\infty} |\lambda|^{-iq}\right)^{\frac{1}{q}}\right] \\ &= \|A_m^{-1}\|^{-1} |\lambda|^m \left[1 - \frac{M_p}{\|A_m^{-1}\|^{-1}} \cdot \frac{1}{(|\lambda|^q - 1)^{\frac{1}{q}}}\right] \geq 0, \text{ a contradiction.} \end{aligned}$$

In the lines above, from (3.5) to (3.6) we use the well-known Hölder's inequality.

It follows that $|\lambda| < \left[1 + \left(\frac{M_p}{\|A_m^{-1}\|^{-1}}\right)^q\right]^{\frac{1}{q}}$.

Similarly we have $|\lambda| > \left[\frac{\|A_0^{-1}\|^{-q}}{(M'_p)^q + \|A_0^{-1}\|^{-q}}\right]^{\frac{1}{q}}$. This completes the proof. \square

4. NUMERICAL EXPERIMENTS

We have already established several estimations for eigenvalues of matrix polynomials. It is in general not possible to compare the sharpness of these bounds. We can only compare them in some special cases by numerical examples. In order to get a good comparison throughout practical examples, we use random data in each example. Moreover, we compare the sharpness of our bounds and those given by N.J. Higham and F. Tisseur [8] by computing examples given there. The experiments were performed using MATLAB 7.10.0 (R2010a).

Consider a 5×5 matrix polynomial $P(z)$ of degree $m = 9$ whose coefficient matrices are

$$A_i = 10^{i-3} \text{rand}(5), \quad i = 0, \dots, 8; \quad A_9 = \text{rand}(5),$$

where $\text{rand}(5)$ denotes a 5×5 random matrix from the normal $(0, 1)$ distribution.

The upper bounds obtained by Higham and Tisseur [8] are given in Table 1, while our new upper bounds are given in Table 2.

Lemmas	Values	Comments
2.3 (2.2)	6.2123×10^5	∞ -norm based
2.3 (2.3)	4.5901×10^5	2-norm based
2.5 (2.13)	5.5691×10^5	∞ -norm based
2.6 (2.14)	6.5821×10^5	Ostrowski, $\beta = 3/4$
2.11 (2.18)	4.5806×10^5	2-norm based
3.1	21.107×10^5	Cauchy's theorem applied for P , 2-norm
3.1	4.5711×10^5	Cauchy's theorem applied for P_U , 2-norm
4.1	23.729×10^5	2-norm based

TABLE 1. Higham and Tisseur's upper bounds

Theorems/Corollaries	Values	Comments
3.1, 3.2, 3.3, 3.4, 3.5	4.5711×10^5	applied for P_U , 2-norm based
3.7, 3.9	5.5198×10^5	2-norm based
3.12, 3.15	3.4120×10^5	2-norm based
3.18, 3.19	5.5198×10^5	2-norm based
3.20	4.5711×10^5	applied for P_U , 2-norm
3.21	4.5927×10^5	applied for P_U , $p=q=2$, 2-norm

TABLE 2. New upper bounds

We can compute exactly the maximal moduli of eigenvalues of $P(z)$ is 1.8468×10^5 . Moreover, Corollary 3.12 and Corollary 3.15 give usually the best upper bounds.

The lower bounds obtained by Higham and Tisseur [8] are given in Table 3, while our new lower bounds are given in Table 4.

Lemmas	Values	Comments
2.6	140×10^{-5}	applied for $C_L(\alpha)$ with $\alpha_i = \ L_{10-i}\ _2$, $\beta = 1/4$
3.1	64.057×10^{-5}	Cauchy's theorem applied for P_U , 2-norm

TABLE 3. Higham and Tisseur's lower bounds

We can compute exactly the minimal moduli of eigenvalues of $P(z)$ is 678.746×10^{-5} . Hence, in general, our new lower bounds are worse than those given by Higham and Tisseur.

Theorems	Values	Comments
3.1	64.057×10^{-5}	Cauchy's theorem applied for P_U , 2-norm
3.8	1.7024×10^{-5}	2-norm based
3.10	1.6650×10^{-5}	2-norm based
3.13	0.23446×10^{-5}	2-norm based
3.16	0.23512×10^{-5}	2-norm based

TABLE 4. New lower bounds

REFERENCES

- [1] P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer-Verlag, New York, 1995.
- [2] M. Dehmer, On the location of zeros of complex polynomials, *J. Inequal. Pure Appl. Math.* **7**(1), Art. 26, 2006.
- [3] B. Datt and N. K. Govil, On the location of the zeros of a polynomial, *J. Approx. Theory* **24** (1978), 78-82.
- [4] G. Dirr and H. K. Wimmer, An Eneström-Kakeya theorem for hermitian polynomial matrices, *IEEE Trans. Automat. Control* **52** (2007), 2151–2153.
- [5] R. A. Frazer, W. J. Duncan and A. R. Collar, *Elementary matrices*, 2nd ed., Cambridge Univ. Press, London and New York, 1955.
- [6] I. Gohberg, P. Lancaster and L. Rodman, *Matrix Polynomials*, Academic Press, New York, 1982.
- [7] S. Hamarling, C.J. Munro and F. Tisseur, An algorithm for the complete solution of quadratic eigenvalue problems, *ACM Trans. Math. Softw.* **39**(3), 2013.
- [8] N.J. Higham and F. Tisseur, Bounds for eigenvalues of Matrix Polynomials, *Linear Algebra and Its Applications* **358** (2003), 5-22.
- [9] A. Joyal, G. Labelle and Q. I. Rahman, On the location of zeros of polynomials, *Cand. Math. Bull.* **10** (1967), 53-63.
- [10] P. Lancaster, *Lambda-matrices and vibrating systems*, Pergamon, Oxford, 1966.
- [11] M. Marden, *Geometry of polynomials*, Mathematical Surveys. Amer. Math. Soc., Rhode Island, 3, 1966.
- [12] G.V. Milovanović and Th. M. Rassias, *Inequalities for polynomial zeros*, In: Survey on Classical Inequalities (Th. M. Rassias, ed.), Mathematics and Its Applications, Vol. 517, pp. 165–202, Kluwer, Dordrecht, 2000.
- [13] V. Mehrmann and D. Watkins, Polynomial eigenvalue problems with Hamiltonian structure, *Electron. Trans. Numer. Anal.* **13** (2002), 106–118.
- [14] G. Singh and W. M. Shah, On the Location of Zeros of Polynomials, *Amer. J. Comp. Math.* **1** (1)(2011), 1-10.
- [15] V. Simoncini and F. Perotti, On the Numerical Solution of $(\lambda^2 A + \lambda B + C)x = b$ and Application to Structural Dynamics, *SIAM J. Sci. Comput.* **23**(6)(2006), 1875–1897.
- [16] F. Tisseur and N. J. Higham, Structured pseudospectra for polynomial eigenvalue problems, with applications, *SIAM Journal On Matrix Analysis And Applications* **23** (1)(2001), 187-208.
- [17] F. Tisseur and K. Meerbergen, The quadratic eigenvalue problem, *SIAM Review*, **43**(2)(2001), 235–286.
- [18] L. Zeng and Y. Su, A backward stable algorithm for quadratic eigenvalue problems, *SIAM J. Matrix Anal. Appl.* **35**(2)(2014), 499-516.

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